MATH 155

Prep/Review for Chapter 9 Test

Name Solutions

Use Algebraic Notation AND Show All of Your Work

For our Chapter 9 Test, you should be able to state each of the following convergence/divergence tests, or theorems:

(a)-Convergence of a Geometric Series
(b)-nth Term Test for Divergence
(c)-The Integral Test
(d)-Convergence of p-Series
(e)-Direct Comparison Test
(f)-Limit Comparison Test
(g)-Alternating Series Test
(h)-Ratio Test
(i)-Root Test

1. Determine the convergence or divergence of the following sequences:

(a) $\left\{\frac{n^3}{3^n}\right\}$, (b) $\left\{\frac{3n^2-n+4}{2n^2+1}\right\}$, (c) $\left\{\frac{(n+1)!}{n!}\right\}$, and (d) $\left\{\frac{(-1)^n}{n!}\right\}$.

If the sequence converges, find its limit. If it diverges, so state and EXPLAIN. *(Be careful with your notation, and show your steps clearly.)*

(a) $\left\{\frac{n^3}{3^n}\right\}$ Let $\left\{\frac{n^3}{3^n}\right\} = \{a_n\}$, with $a_n = \frac{n^3}{3^n}$.

Consider $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n^3}{3^n}$

$= \lim_{x\to\infty}\frac{x^3}{3^x}$ ← Indeterminate form $\frac{\infty}{\infty}$

$= \lim_{x\to\infty}\frac{\frac{d}{dx}(x^3)}{\frac{d}{dx}(3^x)}$

$= \lim_{x\to\infty}\frac{3x^2}{(\ln 3)\cdot 3^x}$ Indeterminate form

$= \frac{3}{\ln 3}\cdot\lim_{x\to\infty}\frac{x^2}{3^x}$ ← $\frac{\infty}{\infty}$

$= \frac{3}{\ln 3}\cdot\lim_{x\to\infty}\frac{\frac{d}{dx}(x^2)}{\frac{d}{dx}(3^x)}$

$= \left(\frac{3}{\ln 3}\right)\lim_{x\to\infty}\frac{2x}{(\ln 3)3^x}$ Indeterminate form $\frac{\infty}{\infty}$

$= \frac{6}{(\ln 3)^2}\cdot\lim_{x\to\infty}\frac{x}{3^x}$ ← $\frac{\infty}{\infty}$

$= \frac{6}{(\ln 3)^2}\cdot\lim_{x\to\infty}\frac{\frac{d}{dx}[x]}{\frac{d}{dx}[3^x]}$

$= \frac{6}{(\ln 3)^2}\cdot\lim_{x\to\infty}\frac{1}{(\ln 3)3^x}$

$= \frac{6}{(\ln 3)^3}\cdot\lim_{x\to\infty}\frac{1}{3^x}$

$= \frac{6}{(\ln 3)^3}\cdot 0$

$= 0$

Yes, the sequence $\left\{\frac{n^3}{3^n}\right\}$ converges to 0.

(b) $\left\{\dfrac{3n^2-n+4}{2n^2+1}\right\}$

Let $\left\{\frac{3n^2-n+4}{2n^2+1}\right\} = \{a_n\}$, with $a_n = \frac{3n^2-n+4}{2n^2+1}$.

Consider $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^2-n+4}{2n^2+1}$

$$= \lim_{n\to\infty} \left[\frac{\frac{3n^2-n+4}{1}}{\frac{2n^2+1}{1}}\right] \cdot \left[\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right]$$

$$= \lim_{n\to\infty} \frac{3-\frac{1}{n}+\frac{4}{n^2}}{2+\frac{1}{n^2}}$$

$$= \frac{3-0+0}{2+0}$$

$$= \frac{3}{2}$$

Yes, the sequence $\left\{\frac{3n^2-n+4}{2n^2+1}\right\}$ converges to $\frac{3}{2}$.

(c) $\left\{\dfrac{(n+1)!}{n!}\right\}$

Let $\left\{\frac{(n+1)!}{n!}\right\} = \{a_n\}$, with $a_n = \frac{(n+1)!}{n!}$.

Consider $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(n+1)!}{n+!}$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot n!}{1\cdot n!}$$

$$= \lim_{n\to\infty} (n+1)$$

$$= \infty$$

This result means that the sequence $\left\{\frac{(n+1)!}{n!}\right\}$ diverges.

(d) $\left\{\dfrac{(-1)^n}{n!}\right\}$

Let $\left\{\dfrac{(-1)^n}{n!}\right\} = \{a_n\}$, with $a_n = \dfrac{(-1)^n}{n!}$,

Consider $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{(-1)^n}{n!}$

We know $-1 \le (-1)^n \le 1$ for all $n \ge 0$.

We can see $\dfrac{-1}{n!} \le \dfrac{(-1)^n}{n!} \le \dfrac{1}{n!}$ for all $n \ge 0$.

According to the Squeeze Theorem for Sequences,

we know $\lim\limits_{n\to\infty}\left(\dfrac{-1}{n!}\right) \le \lim\limits_{n\to\infty}\dfrac{(-1)^n}{n!} \le \lim\limits_{n\to\infty}\left(\dfrac{1}{n!}\right)$

$0 \le \lim\limits_{n\to\infty}\dfrac{(-1)^n}{n!} \le 0$

This means that

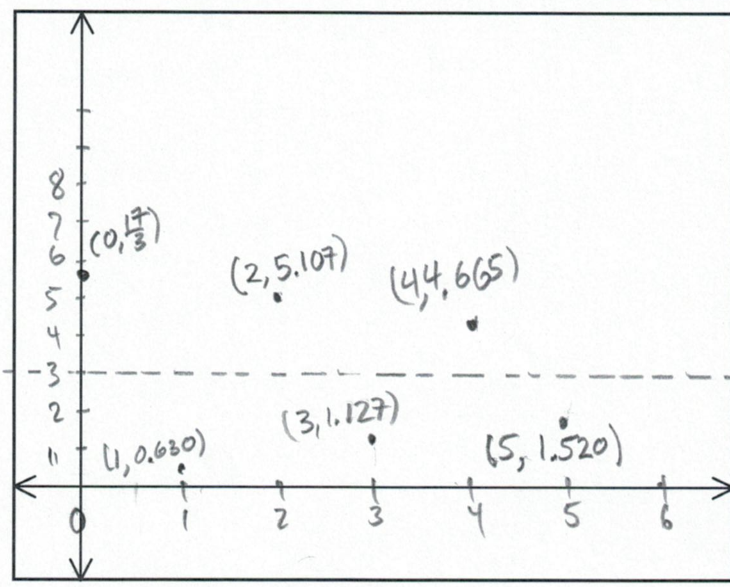
$\lim\limits_{n\to\infty}\dfrac{(-1)^n}{n!} = 0$ and

$\lim\limits_{n\to\infty} a_n = 0$.

Yes, the sequence $\left\{\dfrac{(-1)^n}{n!}\right\}$ converges to 0.

2. **(a)** Find the sum of the series $\sum\limits_{n=0}^{\infty}\dfrac{17}{3}\left(-\dfrac{8}{9}\right)^n$. **(b)** Use a **graphing utility** to find the indicated partial sum S_n and complete the table below. **(c)** Use a graphing utility to graph the first 6 terms of the sequence of partial sums. ***(Label each point.)*** **(d)** Graph the horizontal line that represents this the **sum** of the series and state its equation. *(Be careful with your notation, and show your steps clearly. Round any approximations to the nearest thousandths place.)*

n	0	1	2	3	5	10	20
S_n	$\frac{17}{3}$	0.630	5.107	1.127	1.520	3.821	3.253



HA

$y=3$

$S=3$

$\sum\limits_{n=0}^{\infty}\dfrac{17}{3}\left(\dfrac{-8}{9}\right)^n$, with $a = \dfrac{17}{3}$ and $r = \dfrac{-8}{9}$

$|r| = \left|\dfrac{-8}{9}\right| < 1$ ✓

$= \sum\limits_{n=0}^{\infty} ar^n$

$= \dfrac{a}{1-r}$

$= \dfrac{\left(\frac{17}{3}\right)}{1-\left(\frac{-8}{9}\right)}$

$= \dfrac{\frac{17}{3}}{\frac{9}{9}+\frac{8}{9}}$

$= \dfrac{\frac{17}{3}}{\frac{17}{9}} = \dfrac{17}{3}\cdot\dfrac{9}{17} = 3$

(d) Equation of Horizontal Line: $y=3$

(a) $\sum\limits_{n=0}^{\infty}\dfrac{17}{3}\left(-\dfrac{8}{9}\right)^n = 3$

3. **(a)** Find the sum of the series $\sum_{n=1}^{\infty} \frac{6}{n(n+3)}$. ← Telescoping Series **(b)** Use a **graphing utility** to find the indicated partial sum S_n and complete the table below. **(c)** Use a graphing utility to graph the first 6 terms of the sequence of partial sums. ***(Label each point.)*** **(d)** Graph the horizontal line that represents this the **sum** of the series and state its equation. *(Be careful with your notation, and show your steps clearly. Round any approximations to the nearest thousandths place.)*

n	1	5	10	20	50
S_n	1.5	2.798	3.164	3.394	3.551

Partial Fraction Decomposition

$\frac{6}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3}$

Basic Eqn: $\boxed{6 = A(n+3) + Bn}$

Let $n=0$: $6 = A[(0)+3] + B\cdot(0)$

$6 = 3A$

$\frac{6}{3} = A$

$\boxed{2 = A}$

Let $n=-3$: $6 = 2[(-3)+3] + B(-3)$

$6 = 2\cdot(0) - 3B$

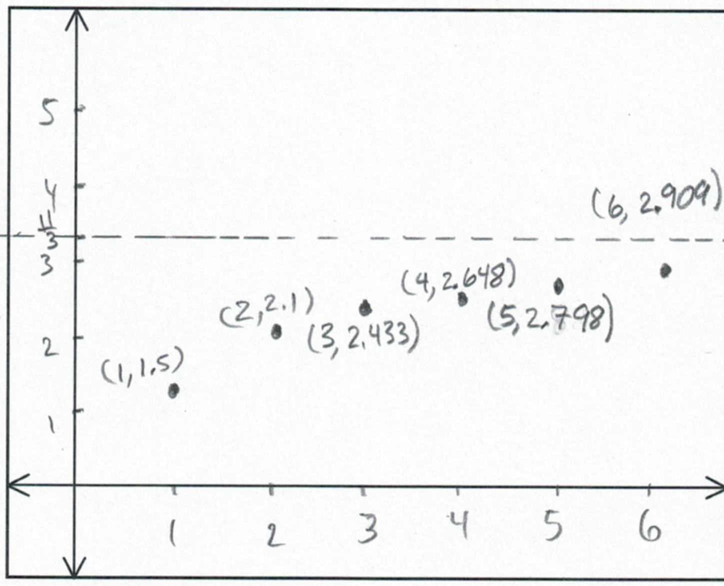
$6 = -3B$

$\frac{6}{-3} = B$

$\boxed{-2 = B}$

+A. $y = \frac{11}{3}$

$S = \frac{11}{3}$



$\sum_{n=1}^{\infty} \frac{6}{n(n+3)} = \sum_{n=1}^{\infty} \left(\frac{2}{n} - \frac{2}{n+3}\right) = 2\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+3}\right)$

$= 2\left[\left(1-\frac{1}{4}\right)+\left(\frac{1}{2}-\frac{1}{5}\right)+\left(\frac{1}{3}-\frac{1}{6}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{5}-\frac{1}{8}\right)+\cdots\right]$,

Don't cancel

Let $b_1 = 2\left(1+\frac{1}{2}+\frac{1}{3}\right)$

$= 2\left(\frac{11}{6}\right)$

$= \frac{11}{3}$

$b_{n+1} = 2\left(\frac{1}{n+1} - \frac{1}{n+4}\right)$

$= \frac{2}{n+1} - \frac{2}{n+4}$

$= \lim_{n\to\infty} S_n$

$= \lim_{n\to\infty} [b_1 - b_{n+1}]$

$= \lim_{n\to\infty} \left[\frac{11}{3} - \left(\frac{2}{n+1} - \frac{2}{n+4}\right)\right]$

$= \frac{11}{3} - 0 + 0$

$= \frac{11}{3}$

(d) Equation of Horizontal Line: $y = \frac{11}{3}$

(a) $\sum_{n=1}^{\infty} \frac{6}{n(n+3)} = \frac{11}{3}$

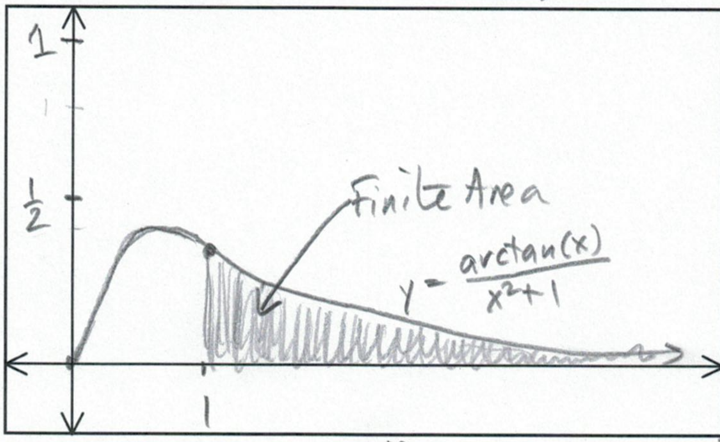
4. State the ***Integral Test***. Use the ***Integral Test*** to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2+1}$. Use a graphing utility to graph $f(x)$, and verify that this graph corresponds with your result from the ***Integral Test***.
(Be careful with your notation, and show your steps clearly.)

$$f'(x) = \frac{(x^2+1)\cdot\left(\frac{1}{x^2+1}\right) - [\arctan(x)](2x)}{(x^2+1)^2}$$

$$f'(x) = \frac{1-2x\arctan(x)}{(x^2+1)^2}$$

(b) Let $f(x) = \frac{\arctan(x)}{x^2+1}$

(c) ***three*** conditions for $f(x)$: for $x \geq 1$, $f(x)$ is positive (I), continuous (II), and decreasing (III)



I $f(x) = \frac{\arctan(x)}{x^2+1} > 0$, for all $x \geq 1$. ✓ (ROPNIP)

II $f(x) = \frac{\arctan(x)}{x^2+1}$ is continuous for all $x \geq 1$, since $x^2+1 \neq 0$, and $\arctan(x)$ is continuous for all $x \geq 1$. (ROCFIC)

III $f(x)$ is decreasing for all $x > 1$, since $f'(x) = \frac{1-2x\arctan(x)}{(x^2+1)^2}$ and $f'(x) < 0$ for all $x > 1$. We can see this by testing $f'(x)$ on $(1, \infty)$ using $x = 2$. $f'(2)$

$$f'(2) = \frac{1-2(2)\arctan(2)}{[(2)^2+1]^2}$$

$$f'(2) \approx \frac{-1.2143}{25} < 0$$

(a) <u>TEST:</u> $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$, either both converge, or both diverge.

Consider $\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{\arctan(x)}{x^2+1}\,dx$

$$= \lim_{b\to\infty} \int_1^b \frac{\arctan(x)}{x^2+1}\,dx$$

$$= \lim_{b\to\infty} \int_{u=\frac{\pi}{4}}^{u=\arctan(b)} \frac{u}{x^2+1}\cdot([x^2+1]\,du)$$

$$= \lim_{b\to\infty} \int_{\frac{\pi}{4}}^{\arctan(b)} u\,du$$

$$= \lim_{b\to\infty} \left[\frac{u^2}{2}\right]_{\frac{\pi}{4}}^{\arctan(b)}$$

$$= \frac{1}{2}\cdot \lim_{b\to\infty} \left([\arctan(b)]^2 - \left[\frac{\pi}{4}\right]^2\right)$$

Let $u = \arctan(x)$

$$\frac{du}{dx} = \frac{1}{x^2+1}$$

$$du = \frac{du}{dx}\cdot dx$$

$$du = \frac{1}{x^2+1}\,dx$$

$$(x^2+1)\,du = dx$$

if $x = 1$
$u = \arctan(1)$
$u = \frac{\pi}{4}$
if $x = b$
$u = \arctan(b)$

$$= \frac{1}{2}\cdot\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right]$$

$$= \frac{1}{2}\cdot\left[\frac{\pi^2}{4} - \frac{\pi^2}{16}\right]$$

$$= \frac{1}{2}\cdot\left[\frac{4\pi^2 - \pi^2}{16}\right]$$

$$= \frac{3\pi^2}{32}$$

This means that $\int_1^{\infty} f(x)\,dx$ converges. The Integral Test tells us that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2+1}$ converges as well.

"Think" $\sum_{n=1}^{\infty} \frac{4^n}{3^n-1} \approx \sum_{n=1}^{\infty} \frac{4^n}{3^n} \approx \sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n \leftarrow |r| = \left|\frac{4}{3}\right| > 1$ Diverges, as a Geometric Series

5. Use the ***Direct Comparison Test*** to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{4^n}{3^n - 1}$. Be sure to show that any conditions for the application of this test are met.

(Be careful with your notation, and show your steps clearly.)

State the ***Direct Comparison Test***: Let $0 < a_n \le b_n$ for all $n \ge 1$.

(I) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(II) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

TEST:

Let $\sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n = \sum_{n=1}^{\infty} a_n$, with $a_n = \left(\frac{4}{3}\right)^n$ and let $\sum_{n=1}^{\infty} \frac{4^n}{3^n-1} = \sum_{n=1}^{\infty} b_n$, with $b_n = \frac{4^n}{3^n-1}$.

We know that $\left(\frac{4}{3}\right)^n > 0$ for all $n \ge 1$, so $a_n > 0$ for all $n \ge 1$, since a ratio of positive numbers is positive. (RoPNiP)

We know $4^n > 0$ and $3^n - 1 \ge 0$ for all $n \ge 1$. We can see that $\frac{4^n}{3^n-1} > 0$ for all $n \ge 1$, since a ratio of positive numbers is positive. (RoPNiP)

This means that $b_n > 0$ for all $n \ge 1$.

We know that $3^n - 1 \le 3^n$ for all $n \ge 1$, and we can use algebra to see the following:

$$\frac{1}{(3^n)(3^n-1)}\left[\frac{3^n-1}{1}\right] \le \frac{1}{(3^n)(3^n-1)}\left[\frac{3^n}{1}\right]$$

$$\frac{4^n}{1} \cdot \frac{1}{3^n} \le \frac{1}{3^n-1} \cdot \frac{4^n}{1}$$

$$\frac{4^n}{3^n} \le \frac{4^n}{3^n-1}$$

$$a_n \le b_n$$

Since $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$ is a divergent geometric series, with $|r| = \left|\frac{4}{3}\right| > 1$, the Direct Comparison Test tells us that $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{4^n}{3^n-1}$ is also a <u>divergent series</u>.

"Think" $\sum_{n=1}^{\infty} \frac{n^2+10}{4n^5+n^3} \approx \sum_{n=1}^{\infty} \frac{n^2}{4n^5} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^3}$ ← convergent p-series $p=3$.

6. Use the ***Limit Comparison Test*** to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n^2+10}{4n^5+n^3}$. Be sure to show that any conditions for the application of this test are met.

(Be careful with your notation, and show your steps clearly.)

State the ***Limit Comparison Test***: Suppose that $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$, where L is finite and positive. Then, the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, either both converge, or both diverge.

TEST:

Let $\sum_{n=1}^{\infty} \frac{n^2+10}{4n^5+n^3} = \sum_{n=1}^{\infty} a_n$, with $a_n = \frac{n^2+10}{4n^5+n^3}$, and let $\sum_{n=1}^{\infty} \frac{1}{4n^3} = \sum_{n=1}^{\infty} b_n$, with $b_n = \frac{1}{4n^3}$. We know that $n^2+10>0$ and $4n^5+n^3>0$ for all $n\geq 1$, and we can see that $\frac{n^2+10}{4n^5+n^3}>0$, since a ratio of positive numbers is positive. (RoPNiP) So, $a_n>0$ for all $n\geq 1$. We know that $1>0$ and $4n^3>0$ for all $n\geq 1$, and we can see that $\frac{1}{4n^3}>0$, since a ratio of positive numbers is positive. (RoPNiP) So, $b_n>0$ for all $n\geq 1$.

Consider $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left[\frac{\frac{n^2+10}{4n^5+n^3}}{\frac{1}{4n^3}}\right]$

$= \lim_{n\to\infty}\left(\frac{n^2+10}{4n^5+n^3}\right)\left(\frac{4n^3}{1}\right)$

$= \lim_{n\to\infty} \frac{4n^5+40n^3}{4n^5+n^3}$

$= \lim_{n\to\infty}\left[\frac{\frac{4n^5+40n^3}{1}}{\frac{4n^5+n^3}{1}}\right]\cdot\left[\frac{\frac{1}{4n^5}}{\frac{1}{4n^5}}\right]$

$= \lim_{n\to\infty} \frac{1+\frac{10}{n^2}}{1+\frac{1}{4n^2}}$

$= \frac{1+0}{1+0}$

$= \underline{1}$ ← Finite & Positive

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{4n^3} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a convergent p-series, with $p=3$, the Limit Comparison Test tells us that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n^2+10}{4n^5+n^3}$ converges as well.

7. **(a)** Use the ***Alternating Series Test*** to prove the convergence of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3n+2}$. Be sure to show that any conditions for the application of this test are met. *(Be careful with your notation, and show your steps clearly.)*

State the ***Alternating Series Test***: Let $a_n > 0$. The alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converge if the following two conditions are met:

(I) $\lim_{n\to\infty} a_n = 0$, AND (II) $a_{n+1} \le a_n$, for all $n \ge 1$.

7(a) TEST: Let $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3n+2} = \sum_{n=1}^{\infty}(-1)^{n+1}a_n$, with $a_n = \frac{1}{3n+2}$. We know that $1>0$ and $3n+2>0$ for all $n \ge 1$, and we can see that $\frac{1}{3n+2} > 0$, since a ratio of positive numbers is positive. (ROPNIP) So, $a_n > 0$ for all $n \ge 1$.

(I) Consider $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{3n+2} = 0$. ✓

(II) We know that $3n+2 \le 3n+5$ for all $n \ge 1$ and we can see the following:

$$\frac{1}{(3n+2)(3n+5)}\cdot\left[\frac{3n+2}{1}\right] \le \frac{1}{(3n+2)(3n+5)}\left[\frac{3n+5}{1}\right]$$

$$\frac{1}{3n+5} \le \frac{1}{3n+2}$$

$$\frac{1}{3(n+1)+2} \le \frac{1}{3n+2}$$

$$a_{n+1} \le a_n \quad ✓$$

Therefore, by the Alternating Series Test, we can see that $\sum_{n=1}^{\infty}(-1)^n a_n = \sum_{n=1}^{\infty}\frac{(-1)^n}{3n+2}$ converges.

7(b) Does $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3n+2}$ converge absolutely, or conditionally? *(Use the definition of absolute convergence and show your reasoning clearly.)*

Consider $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{3n+2}\right| = \sum_{n=1}^{\infty}\frac{1}{3n+2}$. This series is a divergent harmonic series. Since $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{3n+2}\right|$ diverges and $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3n+2}$ converges, we say that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3n+2}$ converges conditionally.

Remainder: $S = S_N + R_N$, $|R_N| = |S - S_N|$ & $|R_N| \le a_{N+1}$ ← "First Neglected term"

ERROR in Approximation

8. **(a)** Use the Alternating Series Remainder to determine the number of terms required to approximate the sum of $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$ with an error of less than 0.001.

Let $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} = \sum_{n=0}^{\infty} (-1)^n a_n$, with $a_n = \frac{1}{2^n n!}$. R_N is the alternating series remainder, given an approximation $S_N = \sum_{k=0}^{N} \frac{(-1)^k}{2^k k!}$, using a finite number of terms. Since $|R_N| \le a_{N+1}$, where a_{N+1} is the "first neglected term," we can say that $|R_N| \le a_{N+1} < 0.001$. This inequality tells us that the error in our finite approximation is less than 0.001. Solve for N:

$a_{N+1} < 0.001$

$\frac{1}{2^{(N+1)}(N+1)!} < \frac{1}{1000}$

$1{,}000 < 2^{N+1}(N+1)!$

If $N=4$
$1000 < 2^{(4+1)} \cdot (4+1)!$
$1{,}000 < 2^5 \cdot (5!)$
$1{,}000 < 3{,}840$
TRUE!

If $N=3$,
$1{,}000 < 2^{(3+1)}(3+1)!$
$1{,}000 < (2^4)(4!)$
$1{,}000 < 384$, False

S_4 has five terms since $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$ starts with zero in the index.

(b) Use a graphing utility and your result from part **(a)** to write a **finite sum** that approximates the infinite sum, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$, with an error of less than 0.001. *(Hint: Write out your sum and show at least* ***two steps*** *as you compute this finite approximation.)*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \approx S_4$$
$$\approx \sum_{n=0}^{4} \frac{(-1)^n}{2^n n!}$$
$$\approx \frac{(-1)^0}{2^0 \cdot (0!)} + \frac{(-1)^1}{2^1(1!)} + \frac{(-1)^2}{2^2(2!)} + \frac{(-1)^3}{2^3(3!)} + \frac{(-1)^4}{2^4(4!)}$$
$$\approx \frac{1}{1} - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{384}$$
$$\approx \frac{233}{384}$$

$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \approx$ 0.60677

Remainder: $S = S_N + R_N$, $|R_N| = |S - S_N|$ & $|R_N| \leq a_{N+1}$

9. **(a)** Use the Alternating Series Remainder to determine the number of terms required to approximate the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n4^n}$ with an error of less than 0.001.

Let $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n4^n} = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$, with $a_n = \frac{1}{n4^n}$. R_N is the alternating series remainder, given an approximation $S_N = \sum_{k=1}^{N} \frac{(-1)^{k+1}}{k4^k}$, using a finite number of terms. Since $|R_N| \leq a_{N+1}$, where a_{N+1} is the "first neglected term," we can say that $|R_N| \leq a_{N+1} < 0.001$. This inequality tells us that the error in our finite approximation is less than 0.001. Solve for N:

$a_{N+1} < 0.001$

$\frac{1}{(N+1)4^{(N+1)}} < \frac{1}{1000}$

$1000 < (N+1)4^{N+1}$

If $N = 2$,

$1000 < (2+1)4^{(2+1)}$

$1{,}000 < 3 \cdot 4^3$

$1{,}000 < 192$

False!

If $N = 3$

$1000 < (3+1)4^{(3+1)}$

$1{,}000 < 4 \cdot 4^4$

$1{,}000 < 1{,}024$

TRUE!

S_3 has three terms since $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n4^n}$ starts with one in the index.

(b) Use a graphing utility and your result from part **(a)** to write a **finite sum** that approximates the infinite sum, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n4^n}$, with an error of less than 0.001. *(Hint: Write out your sum and show at least **two steps** as you compute this finite approximation.)*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n4^n} \approx S_3$$

$$\approx \sum_{n=1}^{3} \frac{(-1)^{n+1}}{n4^n}$$

$$\approx \frac{(-1)^{1+1}}{(1)4^{(1)}} + \frac{(-1)^{2+1}}{(2)4^{(2)}} + \frac{(-1)^{3+1}}{(3)4^{(3)}}$$

$$\approx \frac{1}{4} - \frac{1}{32} + \frac{1}{192}$$

$$\approx \frac{43}{192}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n4^n} \approx 0.22396$$

10. Use the ***Ratio Test*** to determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n!}{n3^n}$.

Be sure to show that any conditions for the application of this test are met.
(Be careful with your notation, and show your steps clearly.)

State the ***Ratio Test:*** Let Σa_n be a series with nonzero terms. (I) Σa_n converges absolutely if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$. (II) Σa_n diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty$. (III) The Ratio Test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$.

TEST:

Let $\sum_{n=1}^{\infty}\frac{n!}{n3^n}=\sum_{n=1}^{\infty}a_n$, with $a_n=\frac{n!}{n3^n}$. For all $n\geq 1$, we can see that $a_n\neq 0$, since $\frac{n!}{n3^n}\neq 0$.

Consider $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{\frac{(n+1)!}{(n+1)3^{(n+1)}}}{\frac{n!}{n3^n}}\right|$

$$=\lim_{n\to\infty}\frac{(n+1)!}{(n+1)3^{(n+1)}}\cdot\frac{n3^n}{n!}$$

$$=\lim_{n\to\infty}\frac{(n+1)\cdot n!\cdot n\cdot 3^n}{(n+1)\cdot 3^n\cdot 3^1\cdot n!}$$

$$=\frac{1}{3}\cdot\lim_{n\to\infty} n$$

$$=\frac{1}{3}(\infty)$$

$$=\infty$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty$, the series, $\sum_{n=1}^{\infty}\frac{n!}{n3^n}$, diverges according to the Ratio Test.

11. Use the ***Root Test*** to determine the convergence or divergence of the series $\sum_{n=1}^{\infty}\left(\frac{4n}{5n-3}\right)^n$.

(Be careful with your notation, and show your steps clearly.)

<u>State the **Root Test**:</u> (I) The series $\sum a_n$ converges absolutely when $\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1$.

(II) The series $\sum a_n$ diverges when $\lim_{n\to\infty}\sqrt[n]{|a_n|} > 1$, or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \infty$.

(III) The Root Test is inconclusive when $\sqrt[n]{|a_n|} = 1$.

TEST:

Let $\sum_{n=1}^{\infty}\left(\frac{4n}{5n-3}\right)^n = \sum_{n=1}^{\infty} a_n$, with $a_n = \left(\frac{4n}{5n-3}\right)^n$.

Consider $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\left(\frac{4n}{5n-3}\right)^n\right|}$

$$= \lim_{n\to\infty}\frac{4n}{5n-3}$$

$$= \lim_{n\to\infty}\left[\frac{\frac{4n}{1}}{\frac{5n-3}{1}}\right]\cdot\left[\frac{\frac{1}{n}}{\frac{1}{n}}\right]$$

$$= \lim_{n\to\infty}\frac{4}{5-\frac{3}{n}}$$

$$= \frac{4}{5-0}$$

$$= \frac{4}{5} < 1.$$

Since $\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1$, the series, $\sum_{n=1}^{\infty}\left(\frac{4n}{5n-3}\right)^n$, <u>converges absolutely</u>, according to the Root Test.

12. State the definition of the ***nth Taylor Polynomial*** for the function f with a center at c.

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the n-th Taylor Polynomial for f at c.

13. **(a)** Find the ***Taylor Polynomial*** of degree 3 for the function $f(x)=\sqrt{x}$ with a center at $c=4$. **(b)** Find the ***Taylor Polynomial*** of degree 4 for the function $f(x)=\ln(x)$ with a center at $c=2$. *(Be careful with your notation, and show your steps clearly.)*

(a) $f(x)=\sqrt{x}=x^{\frac{1}{2}}$

$f'(x)=\frac{1}{2}x^{-1/2}$

$f''(x)=-\frac{1}{2}\cdot\frac{1}{2}x^{-3/2}=-\frac{1}{4}x^{-3/2}$

$f'''(x)=-\frac{3}{2}\cdot\left[-\frac{1}{4}x^{-5/2}\right]=\frac{3}{8}x^{-5/2}$

$f(4)=\sqrt{4}=2$

$f'(4)=\frac{1}{2}(4)^{-1/2}=\frac{1}{2\sqrt{4}}=\frac{1}{2(2)}=\frac{1}{4}$

$f''(4)=-\frac{1}{4}(4)^{-3/2}=\frac{-1}{4(4)^{3/2}}=-\frac{1}{4(2)^3}=\frac{-1}{4\cdot 8}=\frac{-1}{32}$

$f'''(4)=\frac{3}{8}(4)^{-5/2}=\frac{3}{8(4)^{5/2}}=\frac{3}{8\cdot(2)^5}=\frac{3}{8\cdot 32}=\frac{3}{256}$

$P_3(x)=f(4)+\frac{f'(4)}{1!}(x-4)^1+\frac{f''(4)}{2!}(x-4)^2+\frac{f'''(4)}{3!}(x-4)^3$

$P_3(x)=2+\frac{\left(\frac{1}{4}\right)}{1}(x-4)+\frac{\left(-\frac{1}{32}\right)}{2}(x-4)^2+\frac{\left(\frac{3}{256}\right)}{6}(x-4)^3$

$P_3(x)=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2+\frac{1}{512}(x-4)^3$

(b) $f(x)=\ln(x)$

$f'(x)=\frac{1}{x}=x^{-1}$

$f''(x)=+x^{-2}=-x^{-2}$

$f'''(x)=-2\cdot(-x^{-3})=2x^{-3}$

$f^{iv}(x)=-3\cdot(2x^{-4})=-6x^{-4}$

$f(2)=\ln(2)$

$f'(2)=\frac{1}{2}$

$f''(2)=-(2)^{-2}=-\frac{1}{2^2}=-\frac{1}{4}$

$f'''(2)=2(2)^{-3}=\frac{2}{2^3}=\frac{1}{4}$

$f^{iv}(2)=-6(2)^{-4}=\frac{-6}{2^4}=-\frac{6}{16}=\frac{-3}{8}$

$P_4(x)=f(2)+\frac{f'(2)}{1!}(x-2)^1+\frac{f''(2)}{2!}(x-2)^2+\frac{f'''(2)}{3!}(x-2)^3+\frac{f^{iv}(2)}{4!}(x-2)^4$

$P_4(x)=\ln(2)+\frac{\left(\frac{1}{2}\right)}{1!}(x-2)+\frac{\left(-\frac{1}{4}\right)}{2!}(x-2)^2+\frac{\left(\frac{1}{4}\right)}{3!}(x-2)^3+\frac{\left(-\frac{3}{8}\right)}{4!}(x-2)^4$

$P_4(x)=\ln(2)+\frac{1}{2}(x-2)-\frac{1}{8}(x-2)^2+\frac{1}{24}(x-2)^3-\frac{1}{64}(x-2)^4$

(a) ***Taylor Polynomial*** of degree 3, $P_3(x)=$ $2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2+\frac{1}{512}(x-4)^3$

(b) ***Taylor Polynomial*** of degree 4, $P_4(x)=$ $\ln(2)+\frac{1}{2}(x-2)-\frac{1}{8}(x-2)^2+\frac{1}{24}(x-2)^3-\frac{1}{64}(x-2)^4$

center is zero, $c=0$

14. **(a)** Find the ***Maclaurin polynomial*** of degree 4 for the function $f(x)=\frac{1}{1+x}$. **(b)** Use this polynomial from part **(a)** to approximate $f(0.1)$? **(c)** Use Taylor's Theorem and a graphing utility to obtain an upper bound for the error of the approximation. *(Be careful with your notation, and show your steps clearly.)*

(a)

$f(x) = \frac{1}{1+x} = (1+x)^{-1}$

$f'(x) = -1\cdot(1+x)^{-2} = -(1+x)^{-2}$

$f''(x) = -2\cdot[-(1+x)^{-3}] = 2(1+x)^{-3}$

$f'''(x) = -3\cdot 2(1+x)^{-4} = -6(1+x)^{-4}$

$f^{IV}(x) = -4\cdot[-6(1+x)^{-5}] = 24(1+x)^{-5}$

$f^{V}(x) = -5\cdot 24(1+x)^{-6} = -120(1+x)^{-6}$

$f(0) = (1+0)^{-1} = 1$

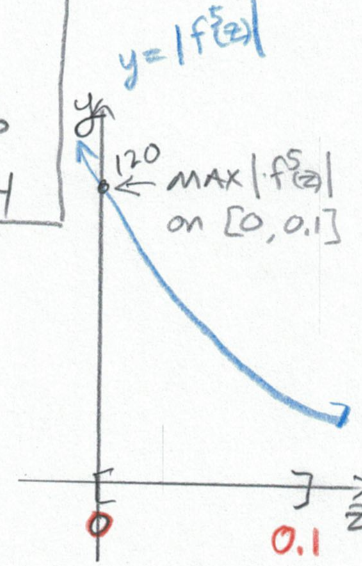
$f'(0) = -(1+0)^{-2} = -1$

$f''(0) = 2(1+0)^{-3} = 2$

$f'''(0) = -6(1+0)^{-4} = -6$

$f^{IV}(0) = 24(1+0)^{-5} = 24$

(c)



$P_4(x) = f(0) + \frac{f'(0)}{1!}(x)^1 + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \frac{f^{IV}(0)}{4!}(x)^4$

$P_4(x) = 1 + \frac{(-1)}{1}(x) + \frac{(2)}{2}(x)^2 + \frac{(-6)}{6}(x)^3 + \frac{(24)}{4!}(x)^4$

$P_4(x) = 1 - x + x^2 - x^3 + x^4$

(b)

$f(0.1) \approx P_4(0.1)$, since $f(x) \approx P_4(x)$.

$f(0.1) \approx 1 - (0.1) + (0.1)^2 - (0.1)^3 + (0.1)^4$

$f(0.1) \approx 0.9091$

The Remainder from Taylor's Theorem is $R_n(x) = \frac{f^{n+1}(z)}{(n+1)!}(x-c)^{n+1}$. So, we know

$R_4(x) = \frac{f^{(5)}(z)}{5!}(x-0)^5$. However, we have $x = 0.1$ and we know that on $[0,0.1]$ there is a $z \in [0,0.1]$ such that

$|R_4(0.1)| \le \frac{(0.1)^5}{5!}\cdot\max|f^5(z)|$. The maximum value of $|f^5(z)|$ on $[0,0.1]$ is 120.

$|R_4(0.1)| \le \frac{(0.1)^5}{5!}\cdot 120$

$|R_4(0.1)| \le (0.1)^5$

$|R_4(0.1)| \le 0.00001$

This means that the error in the $P_4(0.1)$ approximation of $f(0.1)$ is 0.00001, or 1.0×10^{-5}.

(a) ***Maclaurin polynomial*** of degree 4, $P_4(x) =$ $1 - x + x^2 - x^3 + x^4$

(b) $f(0.1) \approx$ $P_4(0.1) \approx 0.9091$

(c) upper bound for the error $\approx$ 0.00001 or 1.0×10^{-5}

center is zero, $c=0$

15. **(a)** Find the ***Maclaurin polynomial*** of degree 5 for the function $f(x)=\sin(x)$. **(b)** Use this polynomial from part **(a)** to approximate $\sin(0.1)$? **(c)** Use Taylor's Theorem and a graphing utility to obtain an upper bound for the error of the approximation. *(Be careful with your notation, and show your steps clearly.)*

(a)

$f(x)=\sin(x)$	$f(0)=\sin(0)=0$
$f'(x)=\cos(x)$	$f'(0)=\cos(0)=1$
$f''(x)=-\sin(x)$	$f''(0)=-\sin(0)=0$
$f'''(x)=-\cos(x)$	$f'''(0)=-\cos(0)=-1$
$f^{IV}(x)=\sin(x)$	$f^{IV}(0)=\sin(0)=0$
$f^{V}(x)=\cos(x)$	$f^{V}(0)=\cos(0)=1$
$f^{VI}(x)=-\sin(x)$	$f^{VI}(0)=-\sin(0)=0$
$f^{VII}(x)=-\cos(x)$	$f^{VII}(0)=-\cos(0)=-1$

$P_5(x)=f(0)+\frac{f'(0)}{1!}(x)^1+\frac{f''(0)}{2!}(x)^2+\frac{f'''(0)}{3!}(x)^3+\frac{f^{IV}(0)}{4!}(x)^4+\frac{f^{V}(0)}{5!}(x)^5$

$P_5(x)=0+\frac{(1)}{1!}(x)^1+\frac{(0)}{2!}(x)^2+\frac{(-1)}{3!}(x)^3+\frac{(0)}{4!}(x)^4+\frac{(1)}{5!}(x)^5$

$P_5(x)=x-\frac{1}{6}x^3+\frac{1}{120}x^5$

(b) $\sin(0.1)\approx P_5(0.1)$, since $\sin(x)\approx P_5(x)$.

$\sin(0.1)\approx(0.1)-\frac{1}{6}(0.1)^3+\frac{1}{120}(0.1)^5$

$\sin(0.1)\approx 0.099833$

(c) The Remainder from Taylor's Theorem is $R_n(x)=\frac{f^{n+1}(z)}{(n+1)!}(x-c)^{n+1}$. So, $R_5(x)=\frac{f^6(z)}{6!}x^6$.

However, we have $x=0.1$ and $c=0$, and on $[0,0.1]$ there is a $z\in[0,0.1]$ such that $|R_5(0.1)|\le\frac{(0.1)^6}{6!}\cdot\max|f^6(z)|$. Since $\max|\pm\cos(z)|=\max|\pm\sin(z)|=1$,

we have $|R_5(0.1)|\le\frac{(0.1)^6}{6!}\cdot 1$

$|R_5(0.1)|\le\frac{0.000001}{720}$

$|R_5(0.1)|\le 1.38889\times10^{-9}$

However, since $f^6(x)=-\sin(x)$, and $f^6(0)=-\sin(0)=0$, we should consider $f^7(x)=-\cos(x)$, because $f^7(0)=-\cos(0)=-1$. This would give as a more accurate upper bound on the error of the approximation.

That is $|R_6(0.1)|\le\frac{(0.1)^7}{7!}\cdot\max|f^7(z)|$ is a more accurate bound on the error.

$|R_6(0.1)|\le\frac{0.0000001}{5040}$

$|R_6(0.1)|\le 1.98413\times10^{-11}$

(a) ***Maclaurin polynomial*** of degree 5, $P_5(x)=$ $x-\frac{1}{6}x^3+\frac{1}{120}x^5$

(b) $\sin(0.1)\approx$ 0.099833

(c) upper bound for the error $\approx$ 1.98413×10^{-11}

16. **(a)** Find the ***radius of convergence*** of the power series $\sum_{n=0}^{\infty} \frac{(x)^n}{2^n}$. ← center is zero, $c=0$.

(b) Find the ***radius of convergence*** of the power series $\sum_{n=0}^{\infty} \frac{(x)^n}{n!}$. ← center is zero, $c=0$.

(Be careful with your notation, and show your steps clearly.)

(a) We can use the Ratio Test to find the radius of convergence for a power series. Let $\sum_{n=0}^{\infty} \frac{(x)^n}{2^n} = \sum_{n=0}^{\infty} u_n$, with $u_n = \frac{x^n}{2^n}$. Since power series converge at their center, we know $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ converges for $x=0$. For the Ratio Test, $u_n = \frac{x^n}{2^n} \neq 0$, so we will not consider $x=0$. Consider $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{2^{n+1}}}{\frac{x^n}{2^n}} \right|$

$= \lim_{n\to\infty} \left|\frac{x}{2}\right|$

$= \left|\frac{x}{2}\right|$

The power series converges absolutely when $\left|\frac{x}{2}\right| < 1$, according to the Ratio Test. So, we'll solve for x to find the radius of convergence.

$2 \cdot \left|\frac{x}{2}\right| < 2 \cdot 1$

$|x| < 2$

$-2 < x < 2$, and the $R = 2$.

(b) Let $\sum_{n=0}^{\infty} \frac{(x)^n}{n!} = \sum_{n=0}^{\infty} u_n$, with $u_n = \frac{(x)^n}{n!}$. For the Ratio Test, $u_n = \frac{x^n}{n!} \neq 0$, so we will not consider $x=0$. Consider $\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right|$

$= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$

$= \lim_{n\to\infty} \left| \frac{x \cdot n!}{(n+1) \cdot n!} \right|$

$= |x| \cdot \lim_{n\to\infty} \frac{1}{n+1}$

$= |x| \cdot 0$

$= 0$

This means the power series converges absolutely for all x values.

$R = \infty$.

(a) ***radius of convergence*** of the power series $\sum_{n=0}^{\infty} \frac{(x)^n}{2^n}$ is $R = 2$

(b) ***radius of convergence*** of the power series $\sum_{n=0}^{\infty} \frac{(x)^n}{n!}$ is $R = \infty$

17. Find the ***interval of convergence*** of the power series $\sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{2^n}$. ← center is -1, $c=-1$. When checking for convergence at the endpoints of the interval, state the series you are testing, **state the name** of your convergence test, and state your result. *(Be careful with your notation, and show your steps clearly.)* We'll use the Ratio Test to find the radius of convergence, and then we might check endpoints for convergence. Let $\sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{2^n} = \sum_{n=1}^{\infty} u_n$, with $u_n = \frac{(-1)^n(x+1)^n}{2^n}$ and we will not consider $x=-1$, since the Ratio Test has $u_n \neq 0$. Consider

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(x+1)^{n+1}}{2^{n+1}}}{\frac{(-1)^n(x+1)^n}{2^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}\cdot 2^n}{2^{n+1}\cdot(x+1)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x+1}{2}\right|$$

$$= \frac{|x+1|}{2}$$

This power series converges absolutely when $\frac{|x+1|}{2} < 1$, according to the Ratio Test. So, we'll solve for x to find the radius of convergence.

$$2\cdot\frac{|x+1|}{2} < 2\cdot 1$$
$$|x+1| < 2$$
$$-2 < x+1 < 2$$
$$-3 < x < 1$$

The radius of convergence is $R=2$, with $c=-1$. Now, we need to test the endpoints for convergence.

Check: $x=-3$

$$\sum_{n=1}^{\infty}\frac{(-1)^n[(-3)+1]^n}{2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n(-2)^n}{2^n}$$
$$= \sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n\cdot 2^n}{2^n}$$
$$= \sum_{n=1}^{\infty}(-1)^{2n} \leftarrow \text{Always even}$$
$$= \sum_{n=1}^{\infty} 1$$

Diverges by the n-th Term Test for Divergence.

Check: $x=1$

$$\sum_{n=1}^{\infty}\frac{(-1)^n[(1)+1]^n}{2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n(2)^n}{2^n}$$
$$= \sum_{n=1}^{\infty}(-1)^n$$

Diverges by the n-th Term Test for Divergence.

Interval of convergence: $(-3,1)$

18. Find the ***interval of convergence*** of the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n}$. When checking for convergence at the endpoints of the interval, state the series you are testing, **state the name** of your convergence test, and state your result. *(Be careful with your notation, and show your steps clearly.)*

center is 5, $c=5$

We'll use the Ratio Test to find the interval of convergence, and then we might check endpoints for convergence. Let $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n5^n} = \sum_{n=1}^{\infty} u_n$, with $u_n = \frac{(-1)^{n+1}(x-5)^n}{n5^n}$, and we will not consider $x=5$, since the Ratio Test has $u_n \neq 0$. Consider $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}(x-5)^{n+1}}{(n+1)5^{n+1}}}{\frac{(-1)^{n+1}(x-5)^n}{n5^n}}\right|$

$$= \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}\cdot n\cdot 5^n}{(n+1)\cdot 5^{n+1}\cdot (x-5)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(x-5)\cdot n}{5\cdot(n+1)}\right|$$

$$= \frac{|x-5|}{5}\cdot \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\left(\frac{\frac{1}{n}}{\frac{1}{n}}\right)$$

$$= \frac{|x-5|}{5}\cdot \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)$$

$$= \frac{|x-5|}{5}\cdot 1$$

$$= \frac{|x-5|}{5}$$

This power series converges absolutely when $\frac{|x-5|}{5} < 1$, according to the Ratio Test. So, we'll solve for x to find the interval of convergence.

$$5\cdot\frac{|x-5|}{5} < 5\cdot 1$$

$$|x-5| < 5$$

$$-5 < x-5 < 5$$

$$0 < x < 10$$

Now, we need to test the endpoints for convergence.

check: $x=0$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}[(0)-5]^n}{n5^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-5)^n}{n5^n}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot(-1)^n\cdot 5^n}{n5^n}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n}$$

always odd

$$= \sum_{n=1}^{\infty}\frac{-1}{n}$$

$$= -\sum_{n=1}^{\infty}\frac{1}{n}$$

Diverges as a harmonic series.

check: $x=10$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}[(10)-5]^n}{n5^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}(5)^n}{n5^n}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

Converges according to the Alternating Series Test.

Interval of convergence: $(0, 10]$

19. Given function defined by the power series $f(x)=\sum_{n=1}^{\infty}\frac{(x)^n}{n}$, find the following series: **(a)** $\int f(x)dx$ and **(b)** $f'(x)$. *(Be careful with your notation, and show your steps clearly.)*

(a) Find $\int f(x)dx$, given $f(x)=\sum_{n=1}^{\infty}\frac{(x)^n}{n}$.

$$\int f(x)dx = \int \left[\sum_{n=1}^{\infty}\frac{x^n}{n}\right]dx$$

$$= C + \sum_{n=1}^{\infty}\frac{1}{n}\cdot\left[\frac{x^{n+1}}{n+1}\right]$$

$$= \sum_{n=1}^{\infty}\frac{x^{n+1}}{n(n+1)} + C$$

(b) Find $f'(x)$, given $f(x)=\sum_{n=1}^{\infty}\frac{(x)^n}{n}$.

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}\left[\sum_{n=1}^{\infty}\frac{x^n}{n}\right]$$

$$f'(x) = \sum_{n=1}^{\infty}\frac{1}{n}\cdot\left[n\,x^{n-1}\right]$$

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1}$$

(a) Power series for $\int f(x)dx = \sum_{n=1}^{\infty}\frac{x^{n+1}}{n(n+1)} + C$

(b) Power series for $f'(x) = \sum_{n=1}^{\infty} x^{n-1}$

20. Given function defined by the power series $\sum_{n=0}^{\infty} \frac{(x)^n}{3^n}$, find the following series: **(a)** $\int f(x)dx$ and **(b)** $f'(x)$. *(Be careful with your notation, and show your steps clearly.)*

(a) Find $\int f(x)dx$, given $f(x) = \sum_{n=0}^{\infty} \frac{(x)^n}{3^n}$.

$$\int f(x)dx = \int \left[\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n \right] dx$$

$$= \int \left[\sum_{n=0}^{\infty} (u)^n \right] \cdot (3du)$$

$$= 3 \int \left[\sum_{n=0}^{\infty} u^n \right] du$$

$$= 3 \left[C + \sum_{n=0}^{\infty} \frac{u^{n+1}}{n+1} \right]$$

$$= 3 \sum_{n=0}^{\infty} \frac{u^{n+1}}{n+1} + C = 3 \sum_{n=0}^{\infty} \frac{\left(\frac{x}{3}\right)^{n+1}}{n+1} + C$$

$$= 3 \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right) \frac{x^{n+1}}{3^{n+1}} + C = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)3^n} + C$$

Let $u = \frac{x}{3}$

$\frac{du}{dx} = \frac{1}{3}$

$du = \frac{du}{dx} \cdot dx$

$du = \frac{1}{3} dx$

$3du = dx$

(b) Find $f'(x)$, given $f(x) = \sum_{n=0}^{\infty} \frac{(x)^n}{3^n}$.

$$\frac{d}{dx}\left[f(x)\right] = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n \right]$$

$$f'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} (u)^n \right]$$

$$f'(x) = \sum_{n=1}^{\infty} \left(n \cdot u^{n-1}\right) \cdot \frac{du}{dx}$$

$$f'(x) = \sum_{n=1}^{\infty} n \left(\frac{x}{3}\right)^{n-1} \cdot \frac{1}{3}$$

$$f'(x) = \sum_{n=1}^{\infty} \frac{n\, x^{n-1}}{3^{n-1} \cdot 3}$$

$$f'(x) = \sum_{n=1}^{\infty} \frac{n\, x^{n-1}}{3^n}$$

Let $u = \frac{x}{3}$

$\frac{du}{dx} = \frac{1}{3}$

(a) Power series for $\int f(x)dx$ = $\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)3^n} + C$

(b) Power series for $f'(x)$ = $\sum_{n=1}^{\infty} \frac{n\, x^{n-1}}{3^n}$

Use $\frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$, for $0<|r|<1$, sum for a Geometric Series

21. **(a)** Find a power series for the function $f(x)=\frac{1}{2x-5}$, centered at $c=-3$. **(b)** Find the ***interval of convergence*** of this power series. *(Be careful with your notation, and show your steps clearly. Hint: Do **not** check the endpoints of the interval of convergence.)*

(a) Let
$$\boxed{\begin{aligned} z &= x-c \\ z &= x-(-3) \\ z &= x+3, \text{ and } x = z-3 \end{aligned}}$$

So, $f(x)=\frac{1}{2x-5}$ becomes

$$f(z-3)=\frac{1}{2(z-3)-5}$$
$$=\frac{1}{2z-6-5}$$
$$=\frac{1}{2z-11}$$
$$=\frac{1}{-11+2z}$$
$$=\left[\frac{1}{-11+2z}\right]\cdot\left[\frac{\frac{-1}{11}}{\frac{-1}{11}}\right]$$
$$=\frac{\frac{-1}{11}}{1-\frac{2z}{11}} = \frac{a}{1-r}$$

$$\boxed{\begin{aligned} a &= \tfrac{-1}{11} \\ r &= \tfrac{2z}{11} \end{aligned}}$$

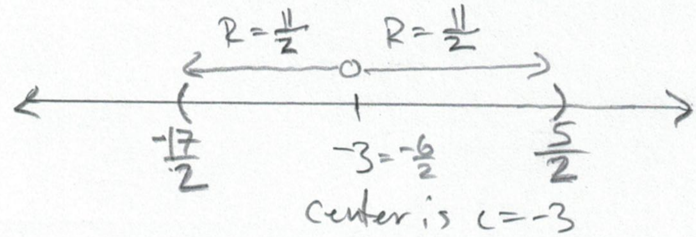
$$=\sum_{n=0}^{\infty} ar^n$$
$$=\sum_{n=0}^{\infty}\left(\frac{-1}{11}\right)\left(\frac{2z}{11}\right)^n$$
$$=\frac{-1}{11}\sum_{n=0}^{\infty}\left(\frac{2}{11}\right)^n z^n$$
$$=\frac{-1}{11}\sum_{n=0}^{\infty}\left(\frac{2}{11}\right)^n (x+3)^n$$

(b) For Geometric Series, we have absolute convergence when $0<|r|<1$. So, to find the interval of convergence we can solve the inequality $|r|<1$.

$$|r|<1$$
$$\left|\frac{2z}{11}\right|<1$$
$$\left|\frac{2(x+3)}{11}\right|<1$$
$$\frac{11}{2}\cdot\left|\frac{2(x+3)}{11}\right|<\frac{11}{2}\cdot 1$$
$$|x+3|<\frac{11}{2}$$
$$-\frac{11}{2}<x+3<\frac{11}{2}$$
$$-\frac{11}{2}-\frac{6}{2}<x+3-3<\frac{11}{2}-\frac{6}{2}$$
$$\frac{-17}{2}<x<\frac{5}{2}$$

$$\left\{x \mid -\frac{17}{2}<x<\frac{5}{2}\right\}=\left(\frac{-17}{2},\frac{5}{2}\right)$$

This means that the interval of convergence is $\left(\frac{-17}{2},\frac{5}{2}\right)$.



"c=-3"

(a) Power series for $\frac{1}{2x-5} = \frac{-1}{11}\sum_{n=0}^{\infty}\left(\frac{2}{11}\right)^n(x+3)^n$, or $\frac{-1}{11}\sum_{n=0}^{\infty}\left[\frac{2(x+3)}{11}\right]^n$

(b) Interval of convergence: $\left(\frac{-17}{2},\frac{5}{2}\right)$

Use $\frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$, for $0<|r|<1$, sum for a Geometric Series

22. **(a)** Find a power series for the function $f(x) = \frac{4}{3x+2}$, centered at $c = 3$. **(b)** Find the ***interval of convergence*** of this power series. *(Be careful with your notation, and show your steps clearly. Hint: Do **not** check the endpoints of the interval of convergence.)*

(a) Let $z = x - c$

$z = x-(3)$

$z = x-3$, and $x = z+3$

So, $f(x) = \frac{4}{3x+2}$ becomes

$$f(z+3) = \frac{4}{3(z+3)+2}$$

$$= \frac{4}{3z+9+2}$$

$$= \frac{4}{3z+11}$$

$$= \frac{4}{11+3z}$$

$$= \left[\frac{4}{11+3z}\right] \cdot \left[\frac{\frac{1}{11}}{\frac{1}{11}}\right]$$

$a = \frac{4}{11}$

$r = \frac{-3z}{11}$

$$= \frac{\frac{4}{11}}{1 + \frac{3z}{11}} = \frac{\frac{4}{11}}{1-\left(\frac{-3z}{11}\right)} = \frac{a}{1-r}$$

$$= \sum_{n=0}^{\infty} ar^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{4}{11}\right)\left(\frac{-3z}{11}\right)^n$$

$$= \frac{4}{11} \sum_{n=0}^{\infty} \left(\frac{-3}{11}\right)^n z^n$$

$$= \frac{4}{11} \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{11}\right)^n (x-3)^n$$

(b) For Geometric Series, we have absolute convergence when $0<|r|<1$. So, to find the interval of convergence we can solve the inequality $|r|<1$.

$$|r|<1$$

$$\left|\frac{-3z}{11}\right|<1$$

$$\left|\frac{3z}{11}\right|<1$$

$$\left|\frac{3(x-3)}{11}\right|<1$$

$$\frac{11}{3} \cdot \left|\frac{3(x-3)}{11}\right| < \frac{11}{3} \cdot 1$$

$$|x-3| < \frac{11}{3}$$

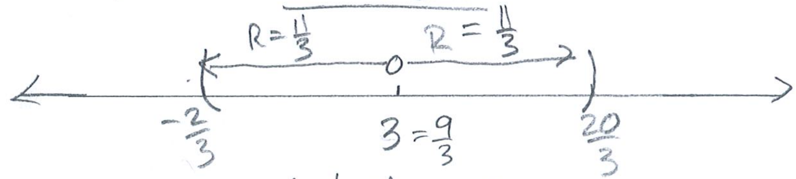
$$-\frac{11}{3} < x-3 < \frac{11}{3}$$

$$-\frac{11}{3}+\frac{9}{3} < x-3+3 < \frac{11}{3}+\frac{9}{3}$$

$$-\frac{2}{3} < x < \frac{20}{3}$$

$$\left\{x \mid -\frac{2}{3}<x<\frac{20}{3}\right\} = \left(-\frac{2}{3}, \frac{20}{3}\right)$$

This means that the interval of convergence is $\left(-\frac{2}{3}, \frac{20}{3}\right)$.



Center is $c=3$

$c=3$

(a) Power series for $\frac{4}{3x+2} = \frac{4}{11} \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{11}\right)^n (x-3)^n$, or $\frac{4}{11} \sum_{n=0}^{\infty} (-1)^n \left[\frac{3(x-3)}{11}\right]^n$

(b) Interval of convergence: $\left(-\frac{2}{3}, \frac{20}{3}\right)$